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Effective Action
of
Spontaneously Broken Gauge Theories

S.-H. Henry Tye
Yan Vtorov-Karevsky

Newman Laboratory of Nuclear Studies
Cornell University, Ithaca, NY 14853

Abstract

The effective action of a Higgs theory should be gauge-invariant. However, the quantum and/or thermal contributions to the effective potential seem to be gauge-dependent, posing a problem for its physical interpretation. In this paper, we identify the source of the problem and argue that in a Higgs theory, perturbative contributions should be evaluated with the Higgs fields in the polar basis, not in the Cartesian basis. Formally, this observation can be made from the derivation of the Higgs theorem, which we provide. We show explicitly that, properly defined, the effective action for the Abelian Higgs theory is gauge invariant to all orders in perturbation expansion when evaluated in the covariant gauge in the polar basis. In particular, the effective potential is gauge invariant. We also show the equivalence between the calculations in the covariant gauge in the polar basis and the unitary gauge. These points are illustrated explicitly with the one-loop calculations of the effective action. With a field redefinition, we obtain the physical effective potential. The $SU(2)$ non-Abelian case is also discussed.

1 Introduction

Effective action plays a very important role in quantum field theory, especially in theories with spontaneous symmetry breakings. For example, we have in mind the inflationary phase (at $T = 0$) or the electroweak phase transition ($T \neq 0$) in the early universe, where space-time dependence plays a prominent role. In these and other important physical applications, the knowledge of the effective action (not just the effective potential) is crucial.

The classical action must be corrected by quantum effects (and by thermal effects at finite temperature). Many calculations of such quantum/thermal effects have been carried out for the effective action [1]. The effective potential term inside the effective action has received the most attention. Since different gauge-fixings were used in the literature, resulting in different formulae for the effective potential term [2], it is not clear which are the appropriate formulae to use.

After the inclusion of quantum effects, the effective action of a Higgs theory must preserve the original global and local symmetries of the classical theory. The effective action must obey the original global symmetry of the theory so that the (would-be) Goldstone bosons will remain massless after spontaneous symmetry breaking. Gauge invariance must be preserved so that the Higgs mechanism can proceed. Using these criteria, we shall re-examine the effective action in various gauges.

In this paper, we shall make the following simple observation. It is an elementary fact that a problem with rotational symmetry should be tackled using polar coordinates, not Cartesian coordinates. The Goldstone modes in Higgs theories should be massless, even when one moves away from the extrema of the effective potential, as is necessary for some physics applications. However, away from the extrema, the would-be Goldstone modes in the Cartesian basis have non-zero masses. It is these masses that bring the gauge dependence into the perturbative calculations of the effective action. As we shall see, the would-be Goldstone bosons in the Cartesian basis are mis-identified. On the other hand, the would-be Goldstone bosons in the polar basis are always massless, at or away from the minimum, as they should be. To correctly determine the effective action of a Higgs theory, perturbation expansion should be carried out in terms of appropriate polar variables. The unitary gauge, and the covariant gauge with the Higgs field in the polar basis are thus appropriate, but the usual covariant and R_ξ gauges [3] are not.

Suppose the Higgs theory has certain global symmetry. It follows that the effective potential will also have that symmetry. After spontaneous symmetry breaking, there are (would-be) Goldstone bosons corresponding to the generators of the broken symmetries. Some of them will be eaten by gauge bosons via the Higgs mechanism. This will happen whenever spontaneous symmetry breaking occurs (i.e., non-zero vacuum expectation values develop), even if we move away from the minimum of the effective potential. So, for the Higgs mechanism to take

place, the Goldstone bosons must remain massless, even away from the minimum of the effective potential. This is true in the polar basis, but not true in the Cartesian basis. These points become more clear in the formal proof of the Higgs theorem, which is presented in Sec. 2. Although there are a number of proofs of the Goldstone theorem [4, 5], to our knowledge, this is the first proof of the Higgs theorem.

The above argument can also be seen in perturbation theory. First, we show that the R_ξ gauge and its variations are poor choices to use in the effective action calculations because the R_ξ gauge-fixing breaks the global symmetry of the theory. As a consequence, the effective potential and other terms in the effective action are no longer globally invariant. Furthermore, mixed propagators cannot be avoided in any practical calculation, as illustrated explicitly by the one-loop correction to the effective potential. This removes the main motivation for the R_ξ gauge. Since the breaking of global symmetry is a consequence of using the R_ξ gauge, it can be avoided if we use another gauge which preserves this symmetry.

Next, let us consider the gauge invariance condition. The key issues are very similar in both the Abelian and the non-Abelian Higgs theories at both zero and finite temperatures. To be specific, let us consider the Abelian Higgs theory (i.e., scalar QED) at zero temperature. The effective action is expected to have the following form:

$$
\begin{aligned}
&\Gamma\,[\,\phi, A_\mu] = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1)\\
&= \int d^{\,4}x \left\{ -V[\phi] \,+\, \frac{1}{2}\, Z_1[\phi]\, \partial_\mu\phi_i\, \partial^\mu\phi_i \,+\, \frac{1}{2}\, Z_2[\phi]\, \frac{\phi_i\phi_j}{\phi^2}\, \partial_\mu\phi_i\, \partial^\mu\phi_j \,- \right.\\
&\left. -\, Z_3[\phi]\, g\epsilon_{ij}\phi_i\partial_\mu\phi_j A^\mu \,+ \frac{1}{2} Z_4[\phi]\, g^2\phi^2 A_\mu A^\mu - \,\frac{1}{4}\, Z_A[\phi]\, F_{\mu\nu}F^{\mu\nu} \,+\, \ldots \right\}
\end{aligned}
$$

where the Higgs field is expressed in Cartesian variables ϕ_i, with $\phi^2 = \phi_1^2+\phi_2^2$. Not displayed are terms that involve higher powers of $A_\mu(x)$ and/or higher derivatives. As the $Z_2[\phi]$-term has two derivatives and is explicitly gauge-invariant, it must be, apriori, included in Eq.(1). When gauge-fixing is necessary, we shall use the covariant gauge, which preserves the global symmetry of the theory. The gauge-fixed effective action $\Gamma_{cov}[\,\phi, A_\mu]$ is now defined as the sum of $\Gamma[\,\phi, A_\mu]$ given in Eq.(1) and the gauge-fixing term

$$
\Gamma_{gf}[\,\phi, A_\mu] = \int d^{\,4}x \left\{ -Z_{gf}[\phi]\, \frac{1}{2\,\xi}\, (\partial_\mu A^\mu)^2 + \ldots \right\} \qquad (2)
$$

Also, any ghost terms will be included in $\Gamma\,[\,\phi, A_\mu]$, although for our purpose they will be absent since we do not introduce ghost background fields. It is this $\Gamma[\phi,\, A_\mu]$, not $\Gamma_{cov}[\phi,\, A_\mu]$, that one should use to construct soliton solutions (magnetic flux, monopole, domain walls etc), to study the evolution of the scalar

field in the inflationary epoch, or the nucleation during the electroweak phase transition, etc. This point is clear at the tree-level already.

Let us consider the classical limit of $\Gamma\,[\,\phi, A_\mu]$ first. In this limit, all $Z_i[\phi] = 1$, except for $Z_2[\phi]$, which is zero. The classical action is invariant under the gauge transformation, with $\phi(x) = \phi_1(x) + i\phi_2(x)$,

$$\begin{aligned} \phi(x) &\rightarrow \phi(x)\, e^{\,i\,\theta(x)} \\ A_\mu(x) &\rightarrow A_\mu(x) + \frac{1}{g}\,\partial_\mu\theta\,(x) \end{aligned} \tag{3}$$

We have argued that $\Gamma\,[\,\phi, A_\mu]$ must be (globally) $U(1)$-invariant and gauge-invariant. This means that all the $Z_i[\phi]$ and $V[\phi]$ are $U(1)$-invariant, and so they are also trivially gauge-invariant. Invariance of $\Gamma\,[\,\phi, A_\mu]$ under the gauge transformation (3) then implies the following conditions:

$$Z_1[\phi] = Z_3[\phi] = Z_4[\phi] \tag{4}$$

If the gauge symmetry condition (4) is not satisfied, the would-be Goldstone boson cannot be gauged away; this would imply a breakdown of the Higgs mechanism. Any $\Gamma\,[\,\phi, A_\mu]$ that is invariant under the gauge transformation (3) will be referred to as actively gauge-invariant. If all the $Z_i[\phi]$ and $V[\phi]$ do not depend on the gauge parameter ξ, $\Gamma\,[\,\phi, A_\mu]$ will be referred to as passively gauge-invariant. $\Gamma\,[\,\phi, A_\mu]$ is gauge-invariant if it is both actively and passively gauge-invariant.

Let us consider $\Gamma\,[\,\phi, A_\mu]$ in the covariant gauge. Carrying out perturbation expansion in the Cartesian (ϕ_1, ϕ_2) basis (the usual approach), a number of observations can be made on the one-loop contribution to $\Gamma\,[\,\phi, A_\mu]$. After separating the gauge-fixing term (2), we get $\Gamma\,[\,\phi, A_\mu]$ as given in Eq.(1) with the following properties:

(1) All the $Z_i[\phi]$ and $V[\phi]$ are globally $U(1)$-invariant, as expected.

(2) All the $Z_i[\phi]$ and $V[\phi]$ explicitly depend on the gauge parameter ξ.

(3) The gauge symmetry condition (4) is satisfied.

(4) $Z_2[\phi]$ is not zero.

Thus $\Gamma\,[\,\phi, A_\mu]$ obtained in the (ϕ_1, ϕ_2) basis is actively but not passively gauge-invariant, that is, it is explicitly ξ–dependent. What is the resolution? It is well-known that the ξ–dependence drops out at the extrema of $V[\phi]$. However, for physics that depends on the shape of the effective potential, one should look at the extrema of the effective action $\Gamma\,[\,\phi, A_\mu]$. Such physics must depend not only on $V[\phi]$, but also on the derivative terms in $\Gamma\,[\,\phi, A_\mu]$. What if the ξ-dependence in $V[\phi]$ is always cancelled by the ξ-dependence coming from the derivative terms, so that all physically measurable quantities are gauge invariant? Can this be the case ? Knowing the $Z_i[\phi]$ and $V[\phi]$ to one-loop, we can check this possibility explicitly. Unfortunately, and somewhat surprisingly, the answer to this question is negative, except in the massless limit (see Sec. 5 and 8).

It is useful to find the source of the ξ-dependence in $\Gamma\,[\,\phi, A_\mu]$. An examination of the $Z_i[\phi]$ and $V[\phi]$ shows that the ξ-dependence comes from the would-be Goldstone boson mass m_G. At the minimum of the effective potential, m_G vanishes and the ξ-dependence drops out. Away from the minimum, the non-vanishing m_G brings in the ξ-dependence. We expect the Higgs mechanism to take place whenever there is a non-zero vacuum expectation value. However, as explained in Sec. 2, the non-vanishing m_G invalidates the Higgs mechanism. This non-vanishing m_G is an artifact of the Cartesian (ϕ_1, ϕ_2) basis. How to circumvent this problem? The solution is suggested by the presence of $Z_2[\phi]$. As $Z_2[\phi]$ is non-zero, the kinetic terms in $\Gamma\,[\,\phi, A_\mu]$ are no longer of the canonical form in the (ϕ_1, ϕ_2) basis. However, in terms of the polar variables (ρ, χ) which are related to the (ϕ_1, ϕ_2) basis through $\phi(x) = \phi_1(x) + i\phi_2(x) = \rho(x)e^{i\chi(x)}$,

$$\frac{\phi_i\phi_j}{\phi^2}\partial_\mu\phi_i\,\partial^\mu\phi_j = (\partial_\mu\rho)^2 \quad \text{and} \quad (\delta_{ij} - \frac{\phi_i\phi_j}{\phi^2})\partial_\mu\phi_i\partial^\mu\phi_j = \rho^2(\partial_\mu\chi)^2. \tag{5}$$

Thus both scalar kinetic terms in $\Gamma\,[\,\phi, A_\mu]$ appear naturally in the polar basis. As the would-be Goldstone boson is always massless in the polar basis, the ξ-dependence problem disappears, and the Higgs theorem always holds (see Sec. 2). These observations clearly suggest that the polar variables should be used in the perturbative calculations of the effective action.

When the one-loop contributions are calculated in the polar gauge (short for the covariant gauge in the polar basis), we find that $\Gamma\,[\,\phi, A_\mu]$ is both passively and actively gauge-invariant, *i.e.* all the $Z_i[\phi]$ and $V[\phi]$ are ξ-independent and Eq.(4) holds. In fact, it is not difficult to show that $\Gamma\,[\,\phi, A_\mu]$ is gauge-invariant to all orders in the perturbation expansion in this gauge, including higher derivative terms. Thus the effective action calculated in the polar basis satisfies the imposed criteria. We may express $\Gamma\,[\,\phi, A_\mu]$ in the polar basis:

$$\begin{aligned}\Gamma[\,\phi, A_\mu] = \int d^4x \Big\{ & -V[\rho] + \frac{1}{2}\,Z_\rho[\rho]\,(\partial_\mu\rho)^2 + \frac{1}{2}\,Z_1[\rho]\,\rho^2\,(\partial_\mu\chi - gA_\mu)^2 - \\ & - \frac{1}{4}\,Z_A[\rho]\,F_{\mu\nu}F^{\mu\nu} + \ldots\Big\}\end{aligned} \tag{6}$$

where Eq.(4) is used and $Z_\rho[\rho] = Z_1[\rho] + Z_2[\rho]$. In this basis, all $Z_i[\rho] = 1$ in the classical limit. With quantum corrections included, $Z_\rho[\rho] \neq Z_1[\rho]$.

With the gauge invariance condition (4) at hand, independent calculations of $Z_1[\rho]$, $Z_3[\rho]$ and $Z_4[\rho]$ are clearly redundant. To avoid this redundancy, a single source term for the combination $B_\mu = A_\mu - \frac{1}{g}\partial_\mu\chi$ can be introduced in the generating functional, instead of separate sources for A_μ and χ. Not surprisingly, this brings us to the unitary gauge. In this gauge, both B_μ and ρ are invariant under the gauge transformation (3), so the source terms for B_μ and ρ are also gauge invariant. Also, it is well-known that only physical degrees of freedom have quantum fluctuations, and no gauge-fixing is necessary in this case. So the

resulting $\Gamma[\phi, A_\mu]$ is by definition gauge-invariant. It is easy to show that the contributions to the effective action in the unitary gauge agree with the covariant gauge calculations in the polar basis to all orders in perturbation expansion. It should be noted that gauge boson propagators in the unitary gauge and in the polar gauge are different (in particular their asymptotic behaviors). So agreement between the two calculations provides a non-trivial check. In the polar gauge, we can see explicitly how the ξ-dependence drops out in $\Gamma[\phi, A_\mu]$. Unitary gauge calculations are more straightforward.

As we have seen, $Z_\rho[\rho]$ is non-trivial in general. To define the physical effective potential, we introduce a field redefinition so that the resulting Z_ρ is 1. In terms of this new Higgs field, the resulting effective potential should be the physical effective potential that measures the energy density in the universe. As an illustration, we show how the Higgs mass is determined from the potential and compare the result to that obtained from standard perturbation theory.

The plan of this paper is the following. In Sec. 2, we give a brief discussion of both the Goldstone and the Higgs theorems. In Sec. 3, we briefly review how to calculate the various $Z_i[\phi]$ and $V[\phi]$ in the effective action. In Sec. 4, we argue why the R_ξ gauge is a poor gauge choice for the determination of $\Gamma[\phi,\, A_\mu]$. To illustrate this point clearly, we review, in Appendix A, the one-loop contribution to the effective potential in the R_ξ gauge for the Abelian Higgs model. In Sec. 5, we give the one-loop contribution to $\Gamma[\phi,\, A_\mu]$ in the covariant gauge in the (ϕ_1, ϕ_2) basis (ignoring four and higher derivative terms). Details of the calculations are given in Appendices B and C. Here we pinpoint the source of the gauge dependence problem to the choice of the (ϕ_1, ϕ_2) basis. We argue that the correct basis should be the polar one. In Sec. 6, gauge-invariance of the effective action to all orders is shown in the polar gauge and some of the one-loop contributions are provided. Appendix D gives the Feynman rules in the polar and unitary gauges. In Sec. 7, we briefly review and comment on the unitary gauge. Its equivalence to the polar gauge to all orders is also shown. In Sec. 8, we discuss the field redefinition needed to obtain a physical effective potential. In Sec. 9, we generalize our earlier discussion to the $SU(2)$ non-Abelian Higgs theory. Sec. 10 contains a brief discussion of the finite temperature case. Sec. 11 contains concluding remarks.

2 Higgs Theorem

Here we present a brief discussion of the Goldstone and the Higgs theorems. Consider the generating functional $Z[J_a]$ of connected diagrams and the corresponding effective action $\Gamma\,[\psi_a]$:

$$\Gamma\,[\psi_a] = Z[J_a] - \int d^{\,4}x\; J_a(x)\,\psi_a(x)\,, \quad \text{where} \quad \frac{\delta\,\Gamma}{\delta\psi_a(x)} = -\,J_a(x) \tag{7}$$

The inverse propagator is given by

$$D^{-1}_{ab}(x-y) = \frac{\delta^2\,\Gamma}{\delta\,\psi_a(x)\,\delta\,\psi_b(y)} \;=\; -\,\frac{\delta\,J_a(x)}{\delta\,\psi_b(y)} \tag{8}$$

Now, variations of the sources can be expressed in terms of the inverse propagators and variations of the fields,

$$\delta\,J_a(x) = -\!\int\! d^{\,4}y\,D^{-1}_{ab}(x-y)\,\delta\,\psi_b(y) \tag{9}$$

Let us consider the Abelian Higgs theory in both the Cartesian and the polar bases. In the Cartesian basis, $\psi_a = (\phi_i,\,A_\mu)$. Under an infinitesimal gauge transformation $\theta(x)$, $\delta\,\psi_a(x)$ become

$$\delta\phi_1 = -\,\theta\,\phi_2\,,\quad \delta\phi_2 = \theta\,\phi_1 \quad \text{and} \quad \delta A_\mu = \frac{1}{g}\,\partial_\mu\theta$$

In the limit $J_a \to 0$, and $\delta J_a \to 0$, spontaneous symmetry breaking takes place when ϕ developes a non-zero vacuum expectation value. We may take $\langle\phi_1\rangle = \upsilon$ and $\langle\phi_2\rangle = 0$. In this case, the Fourier transforms of $\delta\,J_2(x)$ and $\delta\,J_\mu(x)$ are given by

$$D^{-1}_{22}(k)\upsilon\,\theta + \frac{1}{g}\,D^{-1}_{2\,\mu}(k)\,k^\mu\theta \;=\; 0 \tag{10}$$

$$D^{-1}_{\mu\,2}(k)\upsilon\,\theta + \frac{1}{g}\,D^{-1}_{\mu\nu}(k)\,k^\nu\theta \;=\; 0 \tag{11}$$

Note that for the pure complex scalar theory, θ is a non-zero constant and Eq.(10) reduces to

$$D^{-1}_{22}(k)\,\upsilon\,\theta = 0 \tag{12}$$

At zero momentum, this is simply $m_2^2\upsilon\theta = 0$. When $\upsilon \neq 0$, this equation implies that ϕ_2 describes a massless Goldstone boson. This is the proof of the Goldstone theorem due to Jona-Lasinio [5]. Writing $\Gamma\,[\,\phi, A_\mu]$ of the Abelian Higgs theory in the form given in Eq.(1), Eqs.(10, 11) become, for small momenta,

$$Z_1(\upsilon)\,(k^2 - m_2^2)\upsilon\,\theta \;-\; Z_3(\upsilon)\,k^2\upsilon\,\theta = 0 \tag{13}$$

$$Z_3(\upsilon)\,k^\mu g\upsilon^2\,\theta \;-\; Z_4(\upsilon)\,\frac{m_A^2}{g}\,k^\mu\,\theta = 0 \tag{14}$$

where terms with higher powers in momenta have been dropped. Since $\Gamma\,[\,\phi, A_\mu]$ is gauge-invariant, *i.e.*, $Z_1(\upsilon) = Z_3(\upsilon) = Z_4(\upsilon)$, (which follows from the gauge invariance condition (4)), we have $m_A = g\upsilon$ and $m_2 = 0$, as expected. This means the gauge boson is massive when $\upsilon \neq 0$. This is the Higgs theorem. In this (ϕ_1, ϕ_2)–basis, note that both Eq.(10) and Eq.(12) are satisfied only at the minimum (or in general, any extremum) of the potential. This is because

$$m_2^2 = \frac{1}{\upsilon}\frac{\partial V(\upsilon)}{\partial\upsilon}$$

which is zero only at the extremum of the potential.

If we now go to the polar basis, *i.e.* $\psi_a(x) = (\rho, \chi, A_\mu)$, the infinitesimal gauge transformation becomes

$$\delta\rho = 0\,, \quad \delta\chi = \theta \quad \text{and} \quad \delta A_\mu = \frac{1}{g}\,\partial_\mu\theta \tag{15}$$

Following the same analysis, we have

$$D^{-1}_{\chi\chi}(k)\theta + \frac{i}{g}\,D^{-1}_{\chi\mu}(k)\,k^\mu\theta \;=\; 0 \tag{16}$$

$$iD^{-1}_{\mu\chi}(k)\,\theta - \frac{1}{g}\,D^{-1}_{\mu\nu}(k)\,k^\nu\theta \;=\; 0 \tag{17}$$

Since $\Gamma\left[\rho, \chi, A_\mu\right]$ is explicitly $U(1)$–invariant, we have, for any $\upsilon \neq 0$,

$$m^2_\chi = 0, \qquad Z_1(\upsilon) = Z_3(\upsilon) = Z_4(\upsilon) \tag{18}$$

For small momenta, we see that Eq.(16) is always satisfied, at or away from the minimum of the potential. Eq.(17) gives $m^2_A = g^2\upsilon^2$. To obtain the physical gauge boson mass, we must go to the pole of the propagator (sitting at the minimum of the effective potential).

Should we expect the Higgs theorem to hold when υ is not at the minimum of the potential? The answer is definitely *yes*. Imagine one is measuring the energy density (or other physical quantities) when passing through a magnetic flux in a superconductor. Suppose one sits at a point where υ is not zero and not at the minimum of the potential. In this static situation, the energy density is physical and measurable, and the photon is massive. This means the Higgs mechanism must be working, i.e., there is a massless Goldstone boson that has been absorbed by the photon. This physics is correctly captured in the polar basis but not in the Cartesian basis.

In the pure complex scalar theory (i.e., turning off the gauge coupling), the Goldstone theorem holds in the polar basis both at and away from the minimum of the potential. This is not the case in the Cartesian basis. This implies that the proof of the Goldstone theorem should be carried out in the polar basis as well. Although there are many versions of the proof of the Goldstone theorem in the literature, to our knowledge, this is the first formal proof of the Higgs theorem. Using the Lagrangian (70) which will be given in Sec.9, the generalization of the above argument to the $SU(2)$ non-Abelian Higgs theory is straightforward.

3 Preliminaries

Let us briefly review [1] how to calculate quantum corrections to the various $Z_j[\phi]$ and the effective potential $V[\phi]$ inside the effective action $\Gamma[\phi]$. Consider a set of

the scalar fields ϕ_i in some theory. Suppose they fluctuate around some constants υ_i,

$$\phi_i(x) = \upsilon_i + \varphi_i(x) \tag{19}$$

then the effective potential $V[\phi]$ can be Taylor-expanded around υ_i,

$$V[\phi(x)] = V(\upsilon) + \varphi_i(x)\,\partial_i V(\upsilon) + \frac{1}{2}\varphi_i(x)\varphi_j(x)\,\partial_i\partial_j V(\upsilon) + \ldots \tag{20}$$

$$\text{where} \qquad \partial_i V(\upsilon) = \frac{\partial V(\upsilon)}{\partial \upsilon_i}$$

Now we can determine $V[\phi]$ by calculating $V(\upsilon)$, which is given by the one-particle irreducible (1PI)) pure-loop diagrams; alternatively, we can calculate $\partial_i V(\upsilon)$, given by the 1PI tadpole diagrams, and perform an integration. One may also use $\partial_i\partial_j V(\upsilon)$, given by the zero-momentum 1PI two-point functions, and then integrate twice .... This procedure is well-known. In particular, the 1-loop correction $V_1(\upsilon)$ to the effective potential can be calculated using the formula

$$V_1(\upsilon) = -\frac{i}{2}\int \frac{d^4k}{(2\pi)^4} \ln\det\left|iD^{-1}(\upsilon)\right| \tag{21}$$

where $iD^{-1}(\upsilon)$ is the quadratic part of the shifted Lagrangian. Fermionic contributions carry an additional minus sign. In this paper, dimensional regularization is used throughout.

Analogously, one can expand any $Z_j[\phi]$ as

$$Z[\phi(x)] = Z(\upsilon) + \varphi_i(x)\frac{\partial Z(\upsilon)}{\partial \upsilon_i} + \ldots \tag{22}$$

For example, keeping the lowest term after this expansion, the $Z_1[\phi]\,\partial_\mu\phi_i\,\partial^\mu\phi_i$ term in $\Gamma[\phi]$ becomes $Z_1(\upsilon)\,\partial_\mu\varphi_i\,\partial^\mu\varphi_i$. Here $Z_1(\upsilon)$ can be calculated from the appropriate 1PI two-point diagrams, keeping only the terms that are quadratic in external momenta. Of course, the actual perturbative calculations are sometimes complicated by the need for gauge-fixing.

If $V[\phi]$ has a particular global symmetry, the determination of $V[\phi]$ simplifies substantially: all we need to do is 1) choose $\upsilon_1 \neq 0$, $\upsilon_i = 0$, for $i = 2, ..., n$, where these n scalars form a multiplet of the global symmetry; 2) calculate $V(\upsilon_1)$, the pure-loop diagrams, and 3) replace υ_1^2 everywhere inside $V(\upsilon_1)$ by the globally symmetric $\phi^2(x)$ to obtain $V[\phi]$.

However, there is one essential subtlety about the calculations of Z's. There are, in general, two scalar field kinetic terms compatible with the global symmetry, and their coefficients $Z_j[\phi]$ are apriori different. To properly separate $Z_1(\upsilon)$ from $Z_2(\upsilon)$ when these coefficients are calculated from 1PI two-point diagrams, more than one υ_i should be kept non-zero.

4 R_ξ Gauge and Global Symmetry

Suppose the Higgs theory has certain global symmetry that is gauged. The corresponding effective potential $V[\phi]$ must have the same global symmetry as well. After spontaneous symmetry breaking with arbitrary υ, there are (would-be) Goldstone bosons corresponding to the generators of the broken symmetries. The Goldstone bosons must remain massless after the inclusion of quantum effects. Otherwise, the Higgs mechanism is ruined, as was pointed out in Sec. 2. In light of this discussion, R_ξ gauges are clearly unsuitable for the effective action calculations, since they break the global symmetry of the theory. Let us discuss this in more details. We start with the Abelian Higgs Lagrangian (counterterms suppressed),

$$\mathcal{L} = -\frac{1}{4}\, F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\, (D_\mu\phi)^\dagger (D^\mu\phi) - V[\phi] \tag{23}$$

$$\text{where} \qquad D^\mu = \partial^\mu - igA^\mu\,, \qquad V[\phi] = \frac{m^2}{2}\,\phi^2 + \frac{\lambda}{4!}\,\phi^4$$

Introducing the fluctuating fields via Eq.(19), we impose the R_ξ gauge-fixing

$$\mathcal{L}_{gf} = -\frac{1}{2\,\xi}\,(\partial_\mu A^\mu - \xi g\,\epsilon_{ij}\,\varphi_i\,u_j)^2 \tag{24}$$

where u_1 and u_2 are some constants. The 1-loop contribution $V_{R_\xi,1}(\upsilon_i, u_i, \xi)$ to the effective potential was given in Ref.[6] and is reviewed in Appendix A. Here we would like to make just a few remarks on that result:

(a) Since $V_{R_\xi,1}(\upsilon_i, u_i, \xi)$ is not $U(1)$-invariant, we should keep both $\upsilon_1 \neq 0$ and $\upsilon_2 \neq 0$ in our calculations. If we set $\upsilon_2 = u_2 = 0$ (as done in [7]) and calculate $V_{R_\xi,1}(\upsilon_1, 0, u_1, 0, \xi)$, it will not determine the υ_2 dependency in $V_{R_\xi}(\upsilon_1, \upsilon_2, u_1, 0, \xi)$. To find this dependency, one has to evaluate $\partial_2^2 V$, $\partial_2^4 V$ etc., – the $2n$-point diagrams with external ϕ_2's. These calculations are quite involved.

(b) To obtain $V_{R_\xi}[\phi_i, u_i, \xi]$ from $V_{R_\xi}(\upsilon_i, u_i, \xi)$, all υ_i are elevated to ϕ_i, while u_i's should be left untouched. If one wants to avoid mixed propagators in the Feynman diagram calculations (this was the original motivation for the R_ξ gauge), one sets $u_i = \upsilon_i$ (as done in [8], where $u_2 = \upsilon_2 = 0$). This prescription leaves one with an ambiguous $V_{R_\xi}(\upsilon_i, \xi)$ with some υ_i's being part of the scalar field and thus elevated to ϕ_i and some not. To find out which υ_i in $V_{R_\xi}(\upsilon_i, \xi)$ should be elevated to ϕ_i one must evaluate higher-point diagrams such as $\partial_i V$ etc. This calculation is clearly quite involved. In practice, it is a lot easier to allow mixed propagators so one may determine $V_{R_\xi}[\phi_i, u_i, \xi]$ via $V_{R_\xi}(\upsilon_i, u_i, \xi)$ alone.

(c) In the background field method [9], the one-loop contribution still breaks the U(1)-invariance.

(d) A way to remove the gauge-dependence in $V_{R_\xi}[\phi_i, u_i, \xi]$ was formally proposed in Ref.[7, 10]. This was explicitly implemented in the one-loop correction in Ref.[11]. However, the result still depends explicitly on the parameter u_i. Since

the gauge-fixing introduces u_i, they should be considered as gauge parameters. In this sense, $V_{R_\xi}[\phi_i, u_i]$ is still gauge-dependent. Even if one does not want to consider u_i as gauge parameters, the resulting effective potential is not globally U(1) invariant; so it is not actively gauge-invariant.

(e) In the modified R_ξ gauge of [12], U(1)-invariance is again broken.

(f) It is clear that if we set $u_i = 0$ (or $\xi = 0$) in Eq.(24), the resulting $V_{R_\xi}(\upsilon_i)$ is $U(1)$ invariant. But then this is simply the covariant gauge result. In general, covariant gauge obeys the global symmetry of the theory.

5 Effective Action in Covariant Gauge

The effective action should maintain both the global and local symmetries of the classical theory. After spontaneous symmetry breaking, we need this gauge symmetry to gauge away the (would-be) Goldstone bosons, *i.e.*, the Higgs theorem is valid even at the quantum level (see Sec. 2). Also, perturbative quantum effects are not expected to destroy the topological properties of gauge theories (e.g., the Aharonov-Bohm effect for $U(1)$ gauge theory and instanton effects for non-Abelian gauge theories). Since gauge invariance is crucial for the topological properties, the effective action after quantum corrections must be gauge-invariant.

Let us turn to the Abelian Higgs model in the covariant gauge. To carry out perturbative calculations, we must introduce a gauge-fixing term with the gauge parameter ξ. Since the effective action is expected to be globally $U(1)$-invariant, we can write the gauge-fixed effective action $\Gamma_{cov}[\phi, A_\mu]$ in the form

$$\Gamma_{cov}[\phi, A_\mu] = \Gamma[\phi, A_\mu] + \Gamma_{gf}[\phi, A_\mu], \tag{25}$$

where the effective action $\Gamma[\phi, A_\mu]$ is given in Eq.(1) and the gauge-fixing term $\Gamma_{gf}[\phi, A_\mu]$ is given in Eq.(2).

The $V[\phi, \xi]$ and $Z_i[\phi, \xi]$ will be calculated perturbatively. The Feynman rules are well known and given in Appendix B for the sake of completeness. There is one point that we need to emphasize. To separate $Z_1[\phi, \xi]$ and $Z_2[\phi, \xi]$, we need to keep both υ_1 and υ_2 non-zero. This leads to mixed scalar propagators (which also mix with the gauge field propagator). As we shall see, the appearance of mixed scalar propagators is a clear signal that the (ϕ_1, ϕ_2) basis is not the appropriate basis.

It is straightforward to evaluate the one-loop contributions. Some of the relevant results are summarized in Appendix C. The effective potential in the covariant gauge can be obtained from that in the R_ξ-gauge by setting $u_i = 0$, *i.e.*, $V[\phi, \xi]$=$V_{R_\xi}[\phi_i, 0, \xi]$. There are two types of derivative terms in $F_{\mu\nu}F^{\mu\nu}$, one of which appears in the gauge-fixing term $(\partial_\mu A^\mu)^2$. This allows us to uniquely separate $Z_A[\phi]$ and $Z_{gf}[\phi]$ to obtain a gauge-invariant term in $\Gamma\,[\,\phi, A_\mu]$. It is easy to see that $Z_{gf}[\phi]$ is non-trivial.

It is not surprising that $V[\phi, \xi]$ and all the $Z_i[\phi, \xi]$ are explicitly $U(1)$-invariant. However, they are all ξ-dependent, so $\Gamma[\phi, A_\mu]$ is passively gauge-dependent. Here we want to point out the two properties claimed in the introduction (the explicit expressions are given in Appendix C):

(1) $\Gamma[\phi, A_\mu]$ is actively gauge-invariant, *i.e.*, it is invariant under the gauge transformation Eq.(3), *i.e.* Eq.(4) is satisfied, $Z_1[\phi, \xi] = Z_3[\phi, \xi] = Z_4[\phi, \xi]$.

(2) $Z_2[\phi, \xi]$ is non-zero; or equivalently, $Z_\rho[\phi, \xi] \neq Z_1[\phi, \xi]$. Note that $Z_2(v, \xi)$ is finite without regularization. Note also that $Z_2(v, \xi)$ has a pole at the minimum of the classical effective potential.

Is it possible that ξ-dependence of $\Gamma\,[\,\phi, A_\mu]$ is harmless ? Physical quantities are determined at the minima of $\Gamma\,[\,\phi, A_\mu]$. Away from the minima of $V[\phi, \xi]$ this means physics must also depend on the $Z_i[\phi, \xi]$. What if the ξ-dependence in $V[\phi, \xi]$ is always cancelled by the ξ- dependence in $Z_i[\phi, \xi]$, so that all physically measurable quantities are gauge invariant? Can this be the case ? Knowing the $Z_i[\phi, \xi]$ and $V[\phi, \xi]$ to one-loop, we can check this possibility explicitly. Unfortunately, and somewhat surprisingly, the answer to this question is negative (except in the massless limit, see Sec. 8). An easy way to check this point is to consider the so-called physical effective potential. Suppose there exist new (dressed) fields ϕ' and A'_μ as functionals of ϕ and A_μ, such that $\Gamma\,[\,\phi', A'_\mu]$ is gauge-invariant (in particular, ξ-independent). Then one may consider $\Gamma\,[\,\phi', A'_\mu]$ to be satisfactory, and blame all the ξ-dependence in $\Gamma\,[\,\phi, A_\mu]$ on the bad choice of field variables. As a check, we can introduce a new (dressed) field ϕ' as a functional of ϕ, so that the ϕ' kinetic term becomes canonical (*i.e.*, $Z_\rho[\phi'] = 1$). It turns out that $V[\phi']$ is still ξ-dependent (except in the massless limit, *i.e.* $m^2 = 0$; we shall come back to this case later). So we conclude that the covariant gauge in the (ϕ_1, ϕ_2) basis is unsuitable for the determination of $\Gamma\,[\,\phi, A_\mu]$.

The problem with the (ϕ_1, ϕ_2)–basis stems from the presence of fictitious degrees of freedom. An examination of $Z_i[\phi,\, \xi]$ and $V[\phi,\, \xi]$ shows that ξ-dependence comes from the would-be Goldstone mass m_G. As shown in Appendices B and C, two fictitious particles with ξ-dependent masses $m_\pm$ exist in the (ϕ_1, ϕ_2) basis:

$$m_+^2 m_-^2 = \xi m_A^2 m_G^2 \qquad \text{and} \qquad m_+^2 + m_-^2 = m_G^2 \tag{26}$$

where $m_G^2 = m^2 + \lambda\phi^2/6$ and $m_A = g^2\phi^2$ is the gauge boson mass. At the minimum (or, more generally, extremum) of the effective potential, m_G^2 vanishes and the ξ-dependence drops out. Away from the minimum, the non-vanishing m_G^2 brings in the masses $m_\pm$ which in turn bring the ξ-dependence into the $Z_i[\phi,\, \xi]$ and $V[\phi,\, \xi]$. As explained in Sec. 2, non-zero m_G^2 invalidates the Higgs mechanism. How can we avoid this problem?

The presence of $Z_2[\phi,\, \xi]$ suggests that the Lagrangian density of the Abelian Higgs theory should be written in the $(\rho,\, \chi)$ basis, where

$$\partial_\mu \chi = \epsilon_{ij}\, \frac{\phi_i \partial_\mu \phi_j}{\phi^2} \qquad \text{and} \qquad \rho = |\phi| \tag{27}$$

The corresponding kinetic terms are given in Eq.(5). This further suggests that perturbative expansion should be carried out in the polar basis. In this basis the ρ–mass is simply the Higgs mass m_H, while the (would-be) Goldstone boson χ is *always* massless even when we move away from the minimum of the potential.

The (naive) Goldstone mass m_G^2 in the (ϕ_1, ϕ_2)–basis can be expressed in terms of polar variables:

$$m_G^2 = \left.\frac{\partial^2 V[\phi]}{\partial \phi_2^2}\right|_{\substack{\phi_1 = v, \\ \phi_2 = 0}} = \left.\frac{1}{\rho}\frac{\partial V[\rho]}{\partial \rho}\right|_{\rho = v} \tag{28}$$

i.e. m_G^2 is proportional to the first derivative of the effective potential. Away from the minimum of the potential, m_G^2 is non-zero and the above problem with ξ-dependent masses $m_\pm$ (26) appears. Since $V[\rho]$ is $U(1)$–invariant at or away from the minimum of the potential, there should always be a massless mode, as is the case in the polar basis. This means that m_G^2 in Eq.(28) is unphysical. Since one has to move away from the minimum in order to determine the effective action, the polar basis is the correct basis to use, not the (ϕ_1, ϕ_2) one. This argument also applies to theories with only global symmetries.

6 Polar Gauge

The $\Gamma\,[\,\phi, A_\mu]$ calculated in the polar basis in perturbation expansion is completely gauge-invariant. First, we calculate explicitly the one-loop contributions to the various terms in Eq.(6) and find them gauge-invariant. Then we prove the complete gauge-invariance of the multi-loop and higher derivative contributions to the $\Gamma\,[\,\phi, A_\mu]$.

Using Eq.(5) and Eq.(27), the Langrangian density of the Abelian Higgs theory in the polar gauge (*i.e.*, the polar basis in the covariant gauge) is

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu\rho)^2 + \frac{1}{2}\rho^2(\partial_\mu\chi)^2 - g\rho^2 A_\mu\partial^\mu\chi + \frac{1}{2}g^2A^2\rho^2 - \\ &\quad - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V[\rho] - \frac{1}{2\,\xi}(\partial_\mu A^\mu)^2 + \rho\,\bar{c}\,c + \bar{f}\partial_\mu\partial^\mu f\end{aligned} \tag{29}$$

Here the Faddeev-Popov (fermionic) ghosts $\bar{c}$ and c come from the Jacobian $\frac{\partial(\phi_1,\phi_2)}{\partial(\rho,\chi)}$ and have no kinetic term. The ghosts $\bar{f}$ and f come from the covariant gauge-fixing. The gauge-fixed effective action is

$$\begin{aligned}\Gamma_{cov}[\phi, A_\mu] &= \int d^4x \Big\{ -V[\rho] + \frac{1}{2}Z_\rho[\rho](\partial_\mu\rho)^2 + \frac{1}{2}Z_1[\rho]\,\rho^2\,(\partial_\mu\chi - gA_\mu)^2 - \\ &\quad - \frac{1}{4}Z_A[\rho]\,F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\,\xi}Z_{gf}[\rho]\,(\partial_\mu A^\mu)^2 + \ldots \Big\}\end{aligned} \tag{30}$$

After the shift $\rho \to \rho + \upsilon$, the inverse $(\chi,\, A)$–propagator becomes:

$$iD^{-1}_{ab}(k) = \begin{pmatrix} \upsilon^2 k^2 & i\, g\, \upsilon^2 k^\nu \\ -i\, g\, \upsilon^2 k^\mu & -\, g^{\mu\nu}(k^2 - m_A^2) + (1 - \frac{1}{\xi})\, k^\mu k^\nu \end{pmatrix} \tag{31}$$

where $a, b = \chi, \mu = \chi, 0, 1, 2, 3$. The Higgs and ghost inverse propagators are trivial. Feynman rules in this gauge are given in Appendix D.

Using Eq.(21) it is easy to evaluate $V_1[\rho]$:

$$\begin{aligned} V_1(\upsilon) \;\; &= \;\; -\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\left\{ \ln\left[-\frac{1}{\xi}\,(k^2 - m_A^2)^3\,(k^2 - m_H^2)\, k^4 \upsilon^2 \right] - \right. \\ & \left. \quad -\; 2\ln\upsilon \;-\; 2\ln(k^2) \right\} \end{aligned} \tag{32}$$

where $m_A^2 = g^2\upsilon^2$, $m_H^2 = m^2 + \frac{1}{2}\lambda\upsilon^2$. Up to irrelevant constants, $V_1(\upsilon)$ is ξ-independent. The $k^4\upsilon^2$ factor from the matter fields is cancelled by the last two terms coming from the ghosts, leaving behind only the physical degrees of freedom. In contrast, in the (ϕ_1, ϕ_2)-basis, $k^4 \;\to\; (k^2 \,-\, m_+^2)(k^2 \,-\, m_-^2)$ which was the source of all problems with that basis.

So we obtain the effective potential:

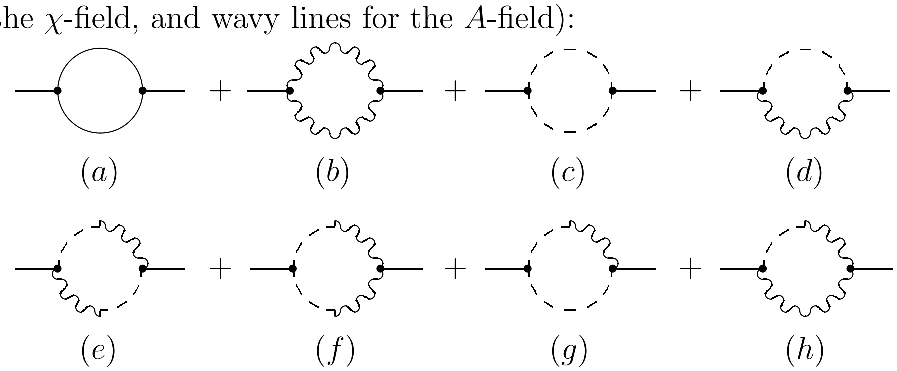
$$V[\rho] \;=\; \frac{m^2\rho^2}{2} \;+\; \frac{\lambda\,\rho^4}{4!} \;+\; \frac{\hbar}{64\pi^2}\left\{ m_H^4(\,\ln\frac{m_H^2}{\mu^2} - \frac{3}{2}) + 3\, m_A^4(\,\ln\frac{m_A^2}{\mu^2} - \frac{5}{6}) \right\} \tag{33}$$

where μ^2 is the renormalization scale.

To evaluate $Z_\rho[\rho]$, let us first consider the 1PI two-point ρ-function $\Sigma(p)$, the p^2-term of which determines $Z_\rho(\upsilon)$. Its one-loop contributions are

$$\begin{aligned} \Sigma'(p) \;\; &= \;\; \frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\left\{ D_H(k)(-i\lambda\upsilon)D_H(p+k)(-i\lambda\upsilon) + \right. \\ & \left. \quad +\; D_{ab}(k)\,V^3_{bc}(k, -(p+k))\,D_{cd}(p+k)\,V^3_{da}(p+k, -k) \right\} \end{aligned} \tag{34}$$

where the prime denotes that p_μ-independent diagrams are omitted. iD_H is the Higgs propagator and iD_{ab} is the $(\chi,\, A)$–propagator (*i.e.* the inverse of iD^{-1}_{ab} given by Eq.(31)). V^3_{ab} are the appropriate 3-point vertices. This equation can be represented by the following diagrams (solid lines for the ρ-field, dashed lines for the χ-field, and wavy lines for the A-field):



(35)

where proper mirror reflections of the above diagrams must be included. Diagram (a) corresponds to the first term of Eq.(34), diagrams $(b-h)$ correspond to the second term. Individual diagrams $(b-h)$ are ξ-dependent. To see that Eq.(34) is ξ-independent, let us note first that the ξ-dependence in Eq.(34) appears only in the $(\chi,\, A)$–propagator iD_{ab}. We may write $D_{ab}(k)$ as

$$D_{ab}(k) = D^{(0)}_{ab}(k) \,+\, \xi\, D^{(1)}_{ab}(k) \tag{36}$$

where

$$D^{(0)}_{ab}(k) = \begin{pmatrix} \dfrac{1}{k^2 \upsilon^2} & 0 \\ 0 & \dfrac{-1}{k^2 - m_A^2}\left(g^{\mu\nu} - \dfrac{k^\mu k^\nu}{k^2}\right) \end{pmatrix} \tag{37}$$

and

$$D^{(1)}_{ab}(k) = \frac{1}{k^4}\begin{pmatrix} -g^2 & igk^\nu \\ -igk^\mu & -\,k^\mu k^\nu \end{pmatrix} \tag{38}$$

$D^{(0)}_{ab}$ is block-diagonal, $D^{(1)}_{ab}(k)$ is hermitian and the ξ-dependence is made explicit. The matrix $V^3_{bc}(k,k')$ of 3-point $(\chi,\, A)$–vertices (*i.e.* $\rho\chi\chi$, $\rho\chi A$ and ρAA) can be easily read from the table in Appendix D:

$$V^3_{bc}(k,k') = 2\, i\, \upsilon \begin{pmatrix} -k\cdot k' & i\, g\, k_\sigma \\ i\, g\, k'_\nu & g_{\nu\,\sigma} g^2 \end{pmatrix} \tag{39}$$

It is straightforward to check that

$$D^{(1)}_{ab}(k)\cdot V^3_{bc}(k,k') = 0 \tag{40}$$

So the ξ-dependence in Eq.(34) drops out and it becomes

$$\Sigma'(p) = \frac{i}{2}\int \frac{d^dk}{(2\pi)^d}\left\{ \frac{\lambda^2 \upsilon^2}{(k^2 - m_H^2)((p+k)^2 - m_H^2)} + \right. \\ \left. + \frac{4g^2 m_A^2}{(k^2 - m_A^2)((p+k)^2 - m_A^2)}\left(d - \frac{k^2}{m_A^2} - \frac{(p+k)^2}{m_A^2} + \frac{[k\cdot(p+k)]^2}{m_A^4}\right)\right\} \tag{41}$$

where d is the space-time dimension. The p^2-term of Eq.(41) gives $Z_\rho(\upsilon)$. In essentially the same fashion, the ξ-dependence in $Z_1(\upsilon)$ also drops out. It is straightforward to check that $Z_1(\upsilon) = Z_3(\upsilon) = Z_4(\upsilon)$. Finally, to one loop we

have

$$
Z_\rho[\rho] = 1 + \frac{\hbar}{16\pi^2}\left\{\frac{\lambda^2\rho^2}{12\,m_H^2} + 3g^2\ln\frac{m_A^2}{\mu^2}\right\} \tag{42}
$$

$$
Z_1[\rho] = 1 + \frac{\hbar}{16\pi^2}\left\{3g^2\frac{m_H^2\ln\dfrac{m_H^2}{\mu^2} - m_A^2\ln\dfrac{m_A^2}{\mu^2}}{m_H^2 - m_A^2} + \frac{m_H^2 - 5m_A^2}{2v^2}\right\} \tag{43}
$$

where μ^2 is the renormalization scale.

Analogously, to evaluate $Z_A[\rho]$ and $Z_{gf}[\rho]$ we consider the 1PI two-point function $\Sigma_A^{\mu\nu}(p)$, extract $g^{\mu\nu}p^2$ and $p^\mu p^\nu$ terms and determine $Z_A[\rho]$ and $Z_{gf}[\rho]$ from them. Again, ξ–dependence drops out and

$$
\Sigma_A^{\mu\nu\,\prime}(p) = -\frac{i}{2}\int\frac{d^dk}{(2\pi)^d}\frac{8g^2(m_A^2 g^{\mu\nu} - k^\mu k^\nu)}{(k^2 - m_A^2)((p+k)^2 - m_H^2)} \tag{44}
$$

(where the prime denotes omission of p^μ–independent diagrams as before).

One readily calculates Z_A:

$$
\begin{aligned}
Z_A[\rho] = 1 + \frac{\hbar}{16\pi^2}\,g^2\Bigg\{&\frac{1}{3}\frac{m_A^2\ln\dfrac{m_A^2}{\mu^2} - m_H^2\ln\dfrac{m_H^2}{\mu^2}}{m_A^2 - m_H^2} - \frac{7m_A^4 + 4m_A^2m_H^2 + m_H^4}{3(m_A^2 - m_H^2)^2} \\
&+ \frac{4m_A^4m_H^2\ln\dfrac{m_A^2}{m_H^2}}{(m_A^2 - m_H^2)^3}\Bigg\}
\end{aligned} \tag{45}
$$

It is also straightforward to extract $Z_{gf}[\rho]$ from Eq.(44). It is obvious that after $\Gamma\,[\,\phi, A_\mu]$ is calculated in the polar basis, it can be re-expressed in *any* basis we want.

Actually, the above ξ-independence is a general feature of the complete $\Gamma\,[\,\phi, A_\mu]$ in perturbation expansion. First, let us examine the reason why the ξ-dependence in the above calculations drops out. As seen in Eq.(40), the product of the ξ-dependent part $D_{ab}^{(1)}(k)$ of the $(\chi,\,A)$-propagator with the vertex matrix $V_{bc}^3(k,k')$ (39) vanishes. As the matrix $V_{bc}^4(k,k')$ of 4-point $(\chi,\,A)$–vertices is proportional to $V_{bc}^3(k,k')$:

$$
V_{bc}^4(k,k') = \frac{1}{v}\,V_{bc}^3(k,k')\;, \tag{46}
$$

we also have

$$
D_{ab}^{(1)}(k)\cdot V_{bc}^4(k,k') = 0 \tag{47}
$$

The only ξ-dependence in the Feynman rules is in the $\xi\, D^{(1)}_{ab}$. Every propagator iD_{ab} (36) in an arbitrary n-point multi-loop diagram contracts with a vertex matrix, either V^3_{bc} or V^4_{bc}. Applying (40) and/or (47) to *any* n-point multi-loop diagram one can readily show its ξ-independence. This means the full effective action, to *all* orders in perturbation expansion including higher derivative terms, is passively gauge-invariant. By construction, $\Gamma\,[\,\phi, A_\mu]$ is also actively gauge-invariant. To conclude, the complete effective action $\Gamma\,[\,\phi, A_\mu]$ evaluated by perturbative expansion in the polar gauge is explicitly gauge-invariant. In the polar gauge, the Goldstone boson χ is always massless, at or away from the minimum of the effective potential. So the Higgs mechanism can take place whenever υ is non-zero.

7 Unitary Gauge

With the gauge invariance condition (4), independent calculations of $Z_1[\rho]$, $Z_3[\rho]$ and $Z_4[\rho]$ are clearly redundant. To avoid this redundancy, let us introduce the combination

$$B_\mu = A_\mu - \frac{1}{g}\partial_\mu \chi \tag{48}$$

which brings us to the unitary gauge Lagrangian :

$$\mathcal{L} \;=\; \frac{1}{2}\,(\partial_\mu \rho)^2 + \frac{1}{2}\,g^2 B^2 \rho^2 - \frac{1}{4}\,\widetilde{F}_{\mu\nu}\widetilde{F}^{\mu\nu} - V[\rho] + \,\rho\,\bar{c}\,c \tag{49}$$

In this case, a single source term for the combination B_μ can be introduced in the generating functional, instead of separate sources for A_μ and χ. In the unitary gauge, both B_μ and ρ are invariant under the gauge transformation (3), so the source terms for B_μ and ρ are gauge invariant. Also, it is well-known that only physical degrees of freedom have quantum fluctuations, and no gauge-fixing is necessary in this case. So the resulting $\Gamma[\phi, A_\mu]$ is by definition gauge-invariant.

Consistency requires that the contributions in the unitary gauge and in the polar gauge must be identical. Before we demonstrate this, let us first consider the one-loop case. The one-loop contribution to the effective potential in the unitary gauge has been calculated long ago [6, 8]:

$$V_1(\upsilon) \;=\; -\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\left\{\ln\left[-g^2\,(k^2-m_A^2)^3\,(k^2-m_H^2)\right] \,-\, 2\ln\upsilon\right\}$$

So $V[\rho]$ in the unitary gauge agrees with $V[\rho]$ in the polar gauge (33).

Evaluation of $\Sigma(p)$ is also straightforward (as before, p_μ-independent terms are omitted):

$$\Sigma'(p) \;=\; \frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\,\{D_H(k)(-i\lambda\upsilon)D_H(p+k)(-i\lambda\upsilon) + \tag{50}$$

$$+\ D^U_{\mu\nu}(k)\,(2\,i g^2 \upsilon\, g_{\nu\rho})\, D^U_{\rho\sigma}(p+k)\,(2\,i g^2 \upsilon\, g_{\sigma\mu})\Big\}$$

where D_H is the Higgs propagator and $D^U_{\mu\nu}$ is the gauge boson propagator (93) in the unitary gauge. It is easy to check that Eq.(50) is exactly Eq.(41), *i.e.* the second term in Eq.(50) reproduces the ξ-independent contributions from diagrams (b, c, d) in Eq.(34). So $Z_\rho[\rho]$ in the unitary gauge agrees with that obtained in the polar gauge. The λ^2 term in $Z_\rho[\rho]$ (42) agrees with that given in Ref.[13]. As expected, $Z_1[\rho]$ is also the same in both gauges. For the evaluation of $Z_A[\rho]$ one readily obtains exactly the same integrand as (44), thus $Z_A[\rho]$ is the same in both gauges as well.

It is straightforward to show the equivalence between the polar gauge and the unitary gauge to any order of perturbation expansion. This must be the case since both gauges yield the physical effective action which is unique. As in the one-loop case, they give the same integrands inside the momentum integrals in any multi-loop Feynman diagram. Here we shall give the key identity and briefly sketch how this equivalence appears in general. The ρ-propagator and its part of the vertices are the same in the two gauges, so we only have to worry about the (χ, A)-propagator and its vertices. Defining the unitary gauge vector-boson propagator matrix as

$$\Delta^U_{ab}(k) = \begin{pmatrix} 0 & 0 \\ & \\ 0 & D^U_{\mu\nu}(k) \end{pmatrix}, \tag{51}$$

it is easy to check the following relation between the polar and the unitary gauges (we use notations introduced in the previous section):

$$V_{ab}\,(k_1, -k_2)\, D_{bc}(k_2)\, V_{cd}\,(k_2, -k_3) \ = \ V_{ab}\,(k_1,\, 0)\, \Delta^U_{bc}(k_2)\, V_{cd}\,(0,\, -k_3) \tag{52}$$

where every V is either $V^{\,3}$ given in Eq.(39) or $V^{\,4}$ given in Eq.(46).

Consider an arbitrary multi-loop Feynman diagram in the polar gauge. Suppose, inside this diagram, it has an internal loop of (χ, A)-lines only (as in Eq.(34)). Using Eqs.(51, 52) and the cyclic property of the trace one can easily see that the following transformation can be done with the integrand:

$$\begin{aligned} &\mathrm{Tr}\,[\, V\,(k_1, -k_2)\, D(k_2)\ \ldots\ D(k_n)\, V\,(k_n, -k_1)\, D(k_1)\,] = \\ &= \mathrm{Tr}\left[\, V\,(0,0)\, \Delta^U(k_2)\ \ldots\ \Delta^U(k_n)\, V\,(0,0)\, \Delta^U(k_1)\,\right] = \\ &= \mathrm{Tr}\left[\, V^U\, D^U(k_2)\ \ldots\ D^U(k_n)\, V^U\, D^U(k_1)\,\right] \end{aligned} \tag{53}$$

where every V^U is either ρAA or $\rho\rho AA$ vertex (same for both gauges) and the trace in the last line is taken over the space-time indices only. So every closed loop of (χ, A)-lines in the polar gauge can be transformed into closed loop of A-lines in the unitary gauge.

After rewriting every closed loop of this type in the unitary gauge, the only remaining (χ, A)-propagators left are the ones that lead to external (χ, A)-fields. In these cases, we have "open chains" of (χ, A)-lines. Each of them can be transformed as

$$\begin{aligned} V_{ab}\,(k_1,-k_2)\,D_{bc}(k_2)\;\ldots\;D_{lm}(k_{n-1})\,V_{mn}\,(k_{n-1},-k_n) = \\ = V_{ab}\,(k_1,0)\,\Delta^U_{bc}(k_2)\;\ldots\;\Delta^U_{lm}(k_{n-1})\,V_{mn}\,(0,-k_n) \end{aligned} \tag{54}$$

Recall that indices $a, b, \ldots$ label χ and A fields together. Once we specify that a and n take the values corresponding to external A_μ field only, we can easily see that

$$\begin{aligned} V_{\mu b}\,(k_1,0)\,\Delta^U_{bc}(k_2)\;\ldots\;\Delta^U_{lm}(k_{n-1})\,V_{m\nu}\,(0,-k_n) = \\ = V^U_{\mu\sigma}\,D^U_{\sigma\rho}(k_2)\;\ldots\;D^U_{\eta\lambda}(k_{n-1})\,V^U_{\lambda\nu} \end{aligned} \tag{55}$$

Thus, polar gauge and unitary gauge calculations yield the same integrands in multi-loop calculations.

It is interesting to compare these two gauges. Gauge-fixing is necessary in the polar gauge; this makes the calculation somewhat more complicated, but it has the advantage that we can explicitly see how the ξ parameter disappears in the determination of $\Gamma[\,\phi, A_\mu]$. In the unitary gauge calculations, there is no gauge-fixing terms to worry about. However, even then, actively gauge-non-invariant terms (such as two-derivative terms involving the gauge fields) will appear in the perturbation expansion. So we have to use the procedure outlined in the introduction to obtain the gauge-invariant $\Gamma[\,\phi, A_\mu]$.

8 Physical Effective Potential

As we have seen, the kinetic terms in the effective action no longer maintain their original canonical forms, *i.e.*, Z_i are not unity. To define the physical effective potential that measures the energy density in the universe, a field redefinition is necessary. The Higgs field in the effective potential should be redefined so that its kinetic term recovers its original canonical form. The effective potential in terms of this new dressed field should be the physical effective potential.

To illustrate this point, let us consider for the moment the Lagrangian of a point particle in classical mechanics. Here the potential $U(x)$ measures the potential energy of the particle. Now, let us consider $x = f(y)$, where $f(y)$ is a well-behaving function. The action can be written as

$$\mathcal{S} = \int dt \left[\frac{\dot{x}^2}{2} - U(x) \right] = \int dt \left[Z(y)\,\frac{\dot{y}^2}{2} - V(y) \right] \tag{56}$$

where $Z(y) = f'(y)^2$. Of course, the theory is the same in either coordinate, although x-coordinate is clearly the natural and most physical choice. In the y-coordinate, we need to know $V(y)$, as well as the function $f(y)$ or, equivalently, $Z(y)$. Starting in the y-coordinate, the potential $U(x)$ cannot be determined from $V(y)$ alone - knowledge of $Z(y)$ is also neccessary. Now let us go back to the Higgs theory case where the situation is completely analogous. A new (dressed) field $\sigma(x)$ as a functional of $\rho(x)$ can be introduced, such that $Z_\sigma[\sigma] = 1$. In the small momenta regime, $U[\sigma]$ determined by $V[\rho]$ and $Z_\rho[\rho]$ is the physical effective potential. One may use it to calculate physical masses, critical temperature and other physical quantities.

Let us introduce the dressed field $\sigma(x)$

$$\sigma[\rho] = \int^{\rho} Z_\rho[\rho']^{\frac{1}{2}}\, d\rho' \tag{57}$$

Now, the physical effective potential is the effective potential expressed in terms of this dressed field $\sigma(x)$:

$$\begin{aligned} U[\sigma] &= V[\rho\,(\sigma)] = \\ &= \frac{m^2\sigma^2}{2} + \frac{\lambda\sigma^4}{4!} + \frac{\hbar}{64\,\pi^2}\Big\{ m_H^4 (\ln\frac{m_H^2}{\mu^2} - \frac{3}{2}) + 3\, m_A^4 (\ln\frac{m_A^2}{\mu^2} - \frac{5}{6}) - \\ &\quad - 6 m_G^2 m_A^2 (\ln\frac{m_A^2}{\mu^2} - 2) + m_G^2 \frac{\lambda\,\sigma}{3}\Big(a\,\mathrm{Arctanh}\frac{\sigma}{a} - \sigma\Big)\Big\} \end{aligned} \tag{58}$$

where

$$m_A^2 = g^2\sigma^2\,, \quad m_H^2 = m^2 + \frac{\lambda\sigma^2}{2}\,, \quad m_G^2 = m^2 + \frac{\lambda\sigma^2}{6} \quad \text{and} \quad a^2 = -\frac{2m^2}{\lambda} \tag{59}$$

For the spontaneous symmetry breaking case, m^2 is negative.

As an illustration, let us see how $U[\sigma]$ is related to the physical Higgs mass. Let the vacuum expectation values be $\langle\rho\rangle = \upsilon$ and $\langle\sigma\rangle = \omega$. Let us first assume that the higher derivative terms in the effective action can be ignored. Then

$$m_H^2 = U''(\omega)\Big|_{\omega\,=\,\omega_{min}} = V''(\upsilon)\cdot Z^{-1}(\upsilon)\Big|_{\upsilon\,=\,\upsilon_{min}} \tag{60}$$

where ω_{min} and υ_{min} are determined by

$$U'(\omega_{min}) = 0 \quad \text{and} \quad V'(\upsilon_{min}) = 0$$

If the Higgs mass is not small, we must include the effects of the higher derivative terms in $\Gamma[\phi, A_\mu]$. Recall that the 1PI Higgs two-point function gives

$$\Sigma(p^2) = V''(\upsilon) - p^2 Z_\rho(\upsilon) - Y(p^2,\, \upsilon) \tag{61}$$

where $Y(p^2, \upsilon) \sim \mathcal{O}(p^4)$ and stands for the relevant higher derivative terms, obtainable from Eq.(41). In the polar gauge, $Y(p^2, \upsilon)$ is gauge-invariant and finite at one-loop. Looking for the Higgs mass as a pole in the propagator in the momentum space, we have

$$Z_\rho(\upsilon)\, p^2 \,+\, Y(p^2,\upsilon) \,-\, V''(\upsilon)\Big|_{\upsilon\,=\,\upsilon_{min}\,,\ p^2\,=\,m_H^2} = 0 \tag{62}$$

or

$$m_H^2 = U''(\omega_{min}) - Y(m_H^2, \upsilon_{min}) \cdot Z^{-1}(\upsilon_{min}) \tag{63}$$

To compare this result with that from the standard perturbation theory, let us introduce the following notations:

$$\begin{aligned} m_H^2 &= m_{H,0}^2 + \hbar\, m_{H,1}^2 + \ldots & \qquad \upsilon_{min} &= \upsilon_0 + \hbar\, \upsilon_1 + \ldots \\ Z_\rho &= 1 + \hbar\, Z_{\rho,1} + \ldots & \qquad \omega_{min} &= \omega_0 + \hbar\, \omega_1 + \ldots \\ V &= V_0 + \hbar\, V_1 + \ldots & & \end{aligned}$$

Recall that the minimum of V_0 is at υ_0 ($\omega_0 = \upsilon_0$ since $Z_{\rho,0} = 1$) *i.e.* $V_0'(\upsilon_0) = 0$. For $m^2 < 0$, we have

$$\upsilon_0^2 = -\frac{6\, m^2}{\lambda} \qquad \text{and} \qquad m_{H,0}^2 = V_0''(\upsilon_0) = -2m^2$$

To order $\hbar$, the minima of $V(\upsilon)$ and $U(\omega)$ are shifted by υ_1 and ω_1 correspondingly, with

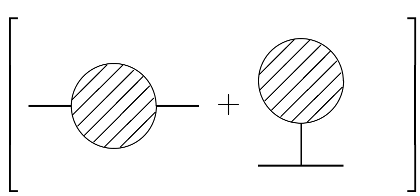
$$\upsilon_1 = -\frac{V_1'(\upsilon_0)}{V_0''(\upsilon_0)} \qquad \text{and} \qquad \omega_1 = \upsilon_1 + \frac{1}{2}\int^{\upsilon_0} Z_1(\upsilon')\, d\upsilon'$$

and we have

$$m_{H,1}^2 = V_1''(\upsilon_0) - m_{H,0}^2 Z_1(\upsilon_0) - Y_1(m_{H,0}^2,\, \upsilon_0) - \frac{V_1'(\upsilon_0)\, V_0'''(\upsilon_0)}{V_0''(\upsilon_0)} \tag{64}$$

or

$$m_{H,1}^2 = i$$



$$\Big|_{\upsilon^2\,=\,\upsilon_0^2\,,\ p^2\,=\,m_{H,0}^2} \tag{65}$$

The last term corresponds to the second diagram : $V_0'''(\upsilon_0)$ is 3-vertex, $-V_0''(\upsilon_0)$ is the inverse scalar propagator at zero momentum and $V_1'(\upsilon_0)$ is clearly the tadpole itself. This is simply the result derived in Ref.[14], where the gauge invariance of $m_{H,1}^2$ is clearly shown.

Let us go back to the massless (*i.e.* $m^2 = 0$) scalar QED case. From Eq.(58), we easily obtain the one-loop (physical) effective potential calculated in the polar basis,

$$U[\sigma] = \frac{\lambda\sigma^4}{4!} + \frac{\hbar}{64\,\pi^2}\left\{\frac{1}{4}\lambda^2 + 3g^2 - \lambda g^2\right\}\ln\frac{\sigma^2}{\mu^2} \tag{66}$$

which is obviously ξ–independent. In the (ϕ_1, ϕ_2)–basis in the covariant gauge, $Z_2[\phi] = 0$ in the limit $m^2 = 0$, and

$$Z_\rho[\phi] = Z_1[\phi] = 1 + \frac{\hbar}{16\,\pi^2}(3-\xi)\,g^2\ln\frac{\phi^2}{\mu^2} \tag{67}$$

and, from Ref.[2],

$$V_{Cartesian}[\phi] = \frac{\lambda\phi^4}{4!} + \frac{\hbar}{64\,\pi^2}\left\{\frac{5}{18}\lambda^2 + 3g^2 - \frac{1}{3}\xi\lambda g^2\right\}\ln\frac{\phi^2}{\mu^2} \tag{68}$$

the resulting physical effective potential is also explicitly ξ–independent,

$$U_{Cartesian}[\sigma] = \frac{\lambda\sigma^4}{4!} + \frac{\hbar}{64\,\pi^2}\left\{\frac{5}{18}\lambda^2 + 3g^2 - \lambda g^2\right\}\ln\frac{\sigma^2}{\mu^2} \tag{69}$$

so we expect all physically measurable quantities to be ξ-independent when calculated in this basis. However, the one-loop (physical) effective potential calculated in the (ϕ_1, ϕ_2)–basis, Eq.(69), differs from that calculated in the polar basis, Eq.(66). Which is correct? Our reasoning clearly implies that the polar gauge calculation, Eq.(66), is correct. First, there is no symmetry associated with the massless limit. When we move away from the massless limit, the ξ-dependence reappears in the (ϕ_1, ϕ_2) basis result, $U_{Cartesian}[\sigma]$. This can be explicitly checked, using the one-loop results for $Z_\rho[\phi]$ and $V[\phi]$ in the covariant gauge. In contrast, Eq.(66) is always ξ-independent. Again, we can trace the difference (the $\lambda^2/36$ term) between the two expressions to the non-zero Goldstone mass m_G^2 in the (ϕ_1, ϕ_2)–basis. As we have argued, this m_G^2 is an artifact.

9 $SU(2)$ Higgs Theories

The above observations generalize to the non-Abelian cases. To be specific, let us consider the $SU(2)$ gauge symmetry case with a $SU(2)$ doublet Higgs field, given by the Lagrangian (we denote $T^a A^{a\,\mu} = A^\mu$, $T^a F^a_{\mu\nu} = F_{\mu\nu}$)

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\,[F_{\mu\nu}F^{\mu\nu}] + \frac{1}{2}\,(D_\mu\phi)^\dagger(D^\mu\phi) - V[\phi] \tag{70}$$

where $\quad D^\mu = \partial^\mu - igA^\mu\,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu\,,\,A_\nu]$

As usual, T^a's are generators of the gauge group. We parametrize the scalar field as

$$\phi(x) = U^{-1}(x)\rho(x) \qquad \text{where} \qquad U(x) = \exp\{-iT^c\chi^c(x)\} \tag{71}$$

The unitary matrix field $U(x)$ contains would-be-Goldstone part of the Higgs sector and the column vector $\rho(x)$ contains remaining scalar fields.

The effective action can be written as

$$\begin{aligned}\Gamma[\,\phi, A^a_\mu] \;&=\; \int d^{\,4}x \left\{ -V[\rho^2] \;+\; \frac{1}{2}\, Z_\rho[\rho^2]\, \partial_\mu\rho^T\partial^\mu\rho \;-\; \frac{1}{2}\, Z_A[\rho^2]\, \mathrm{Tr}\,[F_{\mu\nu}F^{\mu\nu}] \;+ \right. \\ &\left. +\; \frac{1}{2}\, Z_1[\rho^2] g^2\, \rho^T \left(UA^\mu U^{-1} - \frac{i}{g}\,[\partial_\mu U]\, U^{-1} \right)^2 \rho \;+ \ldots \right\}\end{aligned} \tag{72}$$

where $\rho^2 = \rho^T\rho$. Each term in (72) is actively gauge-invariant under the gauge transformation

$$\begin{aligned}\rho(x) \longrightarrow \rho(x) \qquad\qquad & U^{-1}(x) \longrightarrow W(x)U^{-1}(x) \\ A^\mu(x) \longrightarrow W(x)A^\mu(x)W^{-1}(x) - & \frac{i}{g}\,[\partial_\mu W(x)]\, W^{-1}(x)\end{aligned} \tag{73}$$

At the tree-level, $Z_A = Z_1 = Z_\rho = 1$. The easiest way to determine their higher order contributions is to evaluate them in the unitary gauge, where we introduce

$$B^\mu(x) \;=\; U(x)A^\mu(x)U^{-1}(x) - \frac{i}{g}\,[\partial_\mu U(x)]\, U^{-1}(x) \tag{74}$$

For example, the one-loop contribution to the effective potential is

$$V[\rho] = \frac{m^2\rho^2}{2} + \frac{\lambda\,\rho^4}{4!} + \frac{\hbar}{64\,\pi^2}\left\{ m_H^4(\ln\frac{m_H^2}{\mu^2} - \frac{3}{2}) + 9\, m_A^4(\ln\frac{m_A^2}{\mu^2} - \frac{5}{6})\right\} \tag{75}$$

$$Z_\rho[\rho] = 1 + \frac{\hbar}{16\,\pi^2}\left\{ \frac{\lambda^2\rho^2}{12\, m_H^2} + \frac{9\, g^2}{4}\ln\frac{m_A^2}{\mu^2} \right\} \tag{76}$$

where

$$m_H^2 = m^2 + \frac{\lambda\,\rho^2}{2}\,, \qquad m_A^2 = \frac{g^2\rho^2}{4} \tag{77}$$

10 Finite Temperature

By now, it should be obvious that the perturbative contributions to all orders to the finite temperature effective potential $V^\beta[\rho]$ are also gauge-invariant. The $V^\beta[\rho]$ is obtained when we convert the four-momentum integration in Eq.(32)

to a three-momentum integration and a sum over the Matsubara modes (in the imaginary time formalism) [15, 16],

$$\int \frac{d^4k}{(2\pi)^4} \longrightarrow \sum_{\omega} \frac{1}{\beta} \int \frac{d^3k}{(2\pi)^3}\,, \qquad\qquad k_0 \to \omega = 2\pi i\, nT \tag{78}$$

Up to irrelevant constant, the integrand in Eq.(32) is ξ-independent. This means that the $V^{\beta}[\rho]$ is gauge invariant, and is exactly the same as that in the unitary gauge [16]. Clearly, the same gauge invariance argument applies to the various finite temperature $Z_i^{\beta}[\rho]$. Of course, the presence of the heat bath four-velocity introduces additional invariants in the finite temperature effective action [17]. The critical temperature (in the high temperature regime) determined from the unitary gauge was shown to agree with that determined from other gauges [18, 19].

11 Concluding Remarks

It is an elementary fact that a problem with rotational symmetry should be tackled using polar coordinates, not Cartesian coordinates. All Higgs theories have rotational symmetries, so any perturbation expansion should be carried out in the appropriate polar variable basis.

What is wrong with the Cartesian basis? The problem stems from the unphysical degrees of freedom with gauge-dependent masses which are always present in the Cartesian basis and thus bring gauge-dependence into the effective action. These gauge-dependent masses turn to zero when the would-be Goldstone bosons mass vanishes. So, if we carry out the perturbation expansion around the minimum of the effective potential, where the would-be Goldstone bosons are massless, the gauge dependence drops out. This explains why usual gauges in Cartesian basis are acceptable choices for the perturbation expansion around a stable point of the theory.

The would-be Goldstone bosons are always massless in the polar basis, no matter whether we sit at the minimum of the potential or not (and we have to move away from the minimum in order to determine effective action). Physically, the gauge bosons become massive whenever the vacuum expectation value for the scalar field is non-zero. For the Higgs mechanism to work, this requires the would-be Goldstone bosons to be massless, at the minimum or *away* from it. Thus the polar basis should be used for the effective action calculations, while any basis is valid for the perturbative calculations around the minimum of the potential.

Following the above observations, the procedure to obtain the physical $\Gamma[\phi,\, A_\mu]$ in perturbation expansion becomes obvious, for both Abelian and non-Abelian Higgs theories, at zero or finite temperature. First one calculates all terms up to a given loop and up to a given number of derivatives; this gives $\Gamma_{cov}[\phi,\, A_\mu]$. Next one sets all the ghost background fields to zero. Then one removes all terms

that are not actively gauge-invariant. In particular, this removes the terms coming from the gauge-fixing, such as the $Z_{gf}[\phi]$ term. By definition, the resulting $\Gamma[\phi,\, A_\mu]$ is actively gauge-invariant.

When we apply the above procedure to the zeroth order, *i.e.*, the gauge-fixed tree-level effective action, we trivially recover the original classical action. (This is equivalent to the BRST approach. Recall that the gauge-fixed tree-level effective action is BRST invariant. Now the nilpotency of the BRST operator allows us to separate the BRST-closed part $\Gamma[\phi,\, A_\mu]$ from the BRST-exact part (the gauge-fixing terms and the ghosts).) When we apply the above procedure in the general case, the resulting physical $\Gamma[\phi,\, A_\mu]$ obtained in the covariant gauge in the polar basis is gauge invariant to all orders in the perturbation expansion. This is explicitly demonstrated in the one-loop approximation for $\Gamma[\phi,\, A_\mu]$, which agrees with that obtained in the unitary gauge. As is clear from the above discussion, the resulting $\Gamma[\phi,\, A_\mu]$ obtained in the covariant gauge in the Cartesian basis is passively gauge-dependent and hence unphysical.

$\Gamma\,[\,\phi, A_\mu]$ obtained from one-loop calculations in the (ϕ_1, ϕ_2) basis is explicitly ξ-dependent. We have shown that some physics obtained from the minima of $\Gamma\,[\,\phi, A_\mu]$ will also be ξ-dependent. What can we do about this problem ? To remove the explicit ξ-dependence, we have to consider higher-loop contributions. However, from the expressions of the one-loop contributions, the only possibility that the explicit ξ-dependence can be removed is if all-loop contributions are taken into account. This is a daunting project. Since the shape/geometry of the classical potential is given, the relation between the Goldstone modes in the polar and the Cartesian bases is clear. Then one must be able to relate the results in the polar basis to that in the Cartesian basis; or equivalently, the results in the covariant gauge can be converted to that in the unitary gauge. In fact, the mathematical formalism for doing this, namely, the DeWitt-Vilkovisky approach, have been developed [20]. Using this formalism, it has been explicitly shown (at least in the massless case) how the one-loop effective potential in the covariant gauge can be converted to that in the unitary gauge [21]. It will be interesting to check if this approach is applicable in general.

Even in the pure complex scalar theory, the effective potentials obtained from the Cartesian and the polar bases are different. Again, the difference comes from the non-zero Goldstone boson mass in the Cartesian basis. Our argument implies that for theories with only global symmetries, the appropriate polar bases give reliable results.

Acknowledgement

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Appendix A

Here we review [6] the details of the calculation of $V_{R_\xi,1}(\upsilon,u,\xi)$. The quadratic (matter) part of the shifted and gauge-fixed Lagrangian (23) is :

$$iD^{-1}(\upsilon,u,\xi) = \tag{79}$$

$$\begin{pmatrix} \begin{array}{c}(k^2-m_G^2)(\delta_{ij}-\eta_i\,\eta_j)+ \\ +(k^2-m_H^2)\,\eta_i\,\eta_j - M^2(\delta_{ij}-\varepsilon_i\varepsilon_j)\end{array} & i\,g\,\epsilon_{ii'}(\upsilon_{i'}-u_{i'})\,k^\nu \\ & \\ -i\,g\,\epsilon_{jj'}(\upsilon_{j'}-u_{j'})\,k^\mu & \begin{array}{c} -(k^2-m_A^2)\left(g^{\mu\nu}-\dfrac{k^\mu k^\nu}{k^2}\right)- \\ -\dfrac{1}{\xi}\,(k^2-\xi m_A^2)\,\dfrac{k^\mu k^\nu}{k^2}\end{array} \end{pmatrix}$$

where

$$\upsilon^2=\upsilon_1^2+\upsilon_2^2\,, \qquad \eta_i=\frac{\upsilon_i}{\sqrt{\upsilon^2}}\,, \qquad u^2=u_1^2+u_2^2\,, \qquad \varepsilon_i=\frac{u_i}{\sqrt{u^2}} \tag{80}$$

$$m_H^2=m^2+\frac{\lambda\upsilon^2}{2}\,, \quad m_G^2=m^2+\frac{\lambda\upsilon^2}{6}\,, \quad m_A^2=g^2\upsilon^2\,, \quad M^2=\xi\,g^2u^2$$

Using the property

$$\det\begin{vmatrix} A & B \\ C & D\end{vmatrix} = \det|A|\det|D-CA^{-1}B|\,, \qquad (\det|A|\neq 0)\,, \tag{81}$$

the evaluation of $\det|\,iD^{-1}(\upsilon,u,\xi)\,|$ is straightforward :

$$\det|\,iD^{-1}(\upsilon,u,\xi)\,| = (k^2-m_A^2)^3\,(k^2-m_1^2)(k^2-m_2^2)(k^2-m_3^2) \tag{82}$$

where m_i's are defined by :

$$m_1^2+m_2^2+m_3^2 = m_H^2+m_G^2+2\,\xi\,g^2\,u_i\,\upsilon_i \tag{83}$$

$$m_1^2m_2^2+m_1^2m_3^2+m_2^2m_3^2 = m_H^2m_G^2+\xi m_A^2m_G^2+2\,\xi\,g^2m_H^2\,u_i\,\upsilon_i+\xi^2g^4[u_i\,\upsilon_i]^{\,2}$$

$$m_1^2m_2^2m_3^2 = \xi m_A^2m_H^2m_G^2+\xi^2g^4m_H^2[u_i\,\upsilon_i]^2+\xi^2g^4m_G^2\,[\epsilon_{ij}\upsilon_i\,u_j]^{\,2}$$

The result is $V_{R_\xi,1}(\upsilon,u,\xi)$, with all bare quantities replaced by their renormalized counterparts :

$$V_{R_\xi,1}(\upsilon,u,\xi) = \frac{1}{64\pi^2}\left\{3m_A^4\left(\ln\frac{m_A^2}{\mu^2}-\frac{5}{6}\right)+m_1^4\left(\ln\frac{m_1^2}{\mu^2}-\frac{3}{2}\right)+\right. \tag{84}$$

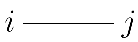
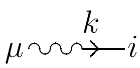
$$\left. +\,m_2^4\left(\ln\frac{m_2^2}{\mu^2}-\frac{3}{2}\right)+m_3^4\left(\ln\frac{m_3^2}{\mu^2}-\frac{3}{2}\right)-2\,m_g^4\left(\ln\frac{m_g^2}{\mu^2}-\frac{3}{2}\right)\right\}$$

where the last term comes from the ghost contribution with $m_g^2=\xi g^2u_i\upsilon_i$.

Appendix B

The Feynman rules of the shifted Abelian Higgs model in the covariant gauge, (ϕ_1, ϕ_2) basis (here we follow the convention of Ref.[22]):

Propagators:

Scalar:		$i\left[\frac{k^2-\xi m_A^2}{D(k^2)}(\delta_{ij}-\eta_i\,\eta_j) \quad + \quad \frac{1}{k^2-m_H^2}\eta_i\,\eta_j\right]$
Gauge:		$-i\left[\frac{1}{k^2-m_A^2}(g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}) \quad + \quad \frac{\xi(k^2-m_G^2)}{D(k^2)}\frac{k^\mu k^\nu}{k^2}\right]$
mixed:		$-\frac{\xi\, g}{D(k^2)}\,k^\mu\,\epsilon_{ij}\,\upsilon_j$

Vertices:

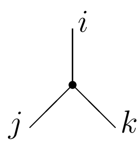
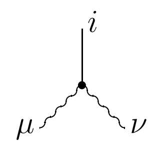
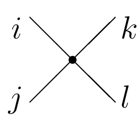
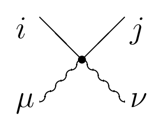
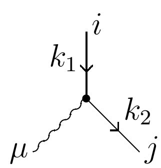
	$-i\,\frac{\lambda}{3}\,(\delta_{ij}\upsilon_k+\delta_{jk}\upsilon_j+\delta_{ki}\upsilon_j)$		$2\,ig^2\upsilon_i\,g^{\mu\nu}$
	$-i\,\frac{\lambda}{3}\,(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$		$2\,ig^2\delta_{ij}\,g^{\mu\nu}$
	$g\epsilon_{ij}\,(k_1+k_2)^{\,\mu}$		

Table 1: Feynman rules in the covariant gauge

where

$$D(k^2) = k^4 - k^2 m_G^2 + \xi m_A^2 m_G^2 = (k^2-m_+^2)(k^2-m_-^2) \tag{85}$$

Appendix C

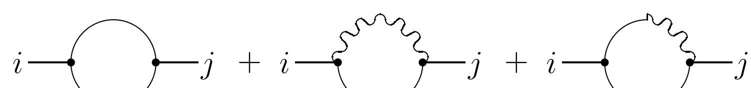
One-loop correction to the effective potential in the covariant gauge, (ϕ_1, ϕ_2)-basis, is simply given by the $u_1 = 0\,,\, u_2 = 0$ limit of Eq.(84):

$$V_{cov,\,1}(v) = \frac{1}{64\pi^2}\left\{3m_A^4\left(\ln\frac{m_A^2}{\mu^2} - \frac{5}{6}\right) + m_H^4\left(\ln\frac{m_H^2}{\mu^2} - \frac{3}{2}\right) + \right. \tag{86}$$

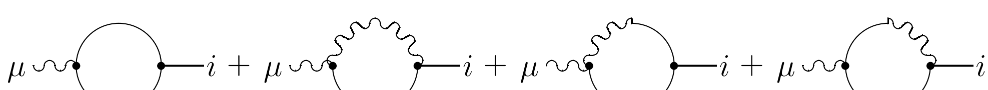
$$\left. + m_+^4\left(\ln\frac{m_+^2}{\mu^2} - \frac{3}{2}\right) + m_-^4\left(\ln\frac{m_-^2}{\mu^2} - \frac{3}{2}\right)\right\}$$

where $m_+^2\,,\, m_-^2$ are defined by Eq.(85)

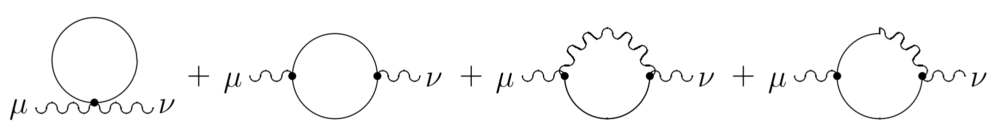
To check the gauge symmetry condition (4) we need to evaluate the following ten diagrams. The p^2 term from diagrams $(a - c)$ contributes to Z_1, the p_μ term from diagrams $(d - g)$ - to Z_3 and $p_\mu = 0$ term in diagrams $(h - k)$ - to Z_4.



(a) (b) (c)



(d) (e) (f) (g)



(h) (i) (j) (k)

Contributions of these diagrams are summarized in the following table:

Following notations were used for the integrals calculated in the minimal substraction scheme :

$$\frac{i}{16\pi^2}B_0(m_1, m_2, p^2) \doteq \mu_0^\epsilon \int \frac{d^{\,d}k}{(2\pi)^d}\,\frac{1}{(k^2 - m_1^2)((k+p)^2 - m_2^2)} \tag{87}$$

$$B_0(m_1, m_2, p^2) = B_0^0(m_1, m_2) + B_0^1(m_1, m_2)\cdot p^2 + O(p^4) + \ldots \tag{88}$$

Shorthand $B(m_1, m_2) = B(1,2)$ is used from now on.

$$B_0^0(1,2) \;=\; \frac{m_1^2\ln\dfrac{m_1^2}{\mu^2} - m_2^2\ln\dfrac{m_2^2}{\mu^2}}{m_2^2 - m_1^2} \tag{89}$$

$a)$	$\dfrac{\lambda^2 \upsilon^2}{9}\left\{D^1 - \xi m_A^2 C^1\right\}$
$b)$	$g^2\Big(\frac{1}{2} - 3B_0^0(A,H)\Big) +$
	$+\,\xi g^2\Big\{D^0 - m_G^2 C^0 + m_G^2(2m_H^2 - \xi m_A^2)C^1 - 2m_H^2 D^1 - \frac{1}{2}\Big\}$
$c)$	$2\cdot \xi g^2 \dfrac{\lambda \upsilon^2}{3}\left\{m_H^2 C^1 - C^0\right\}$
$d)$	$\dfrac{\lambda^2 \upsilon^2}{9}\left\{D^1 - \xi m_A^2 C^1\right\} + \xi g^2 \dfrac{\lambda \upsilon^2}{3} D^1$
$e)$	$g^2\left(\frac{1}{2} - 3B_0^0(A,H)\right) + \xi g^2\left\{m_G^2(m_H^2 - \xi m_A^2)C^1 - m_H^2 D^1 - \frac{1}{2}\right\}$
$f)$	$\xi g^2 \dfrac{\lambda \upsilon^2}{3}\left\{m_H^2 C^1 - C^0 - D^1\right\}$
$g)$	$\xi g^2 \left\{D^0 + \dfrac{\lambda \upsilon^2}{3} D^1 + \xi m_A^2 m_G^2 C^1 + \frac{1}{2}\right\}$
$\left.\begin{matrix} h \\ i \end{matrix}\right\}$	$\dfrac{\lambda^2 \upsilon^2}{9}\left\{D^1 - \xi m_A^2 C^1\right\} + \xi g^2 \left\{2\dfrac{\lambda \upsilon^2}{3} D^1 + \xi m_A^2 m_G^2 C^1\right\}$
$j)$	$g^2\left(\frac{1}{2} - 3B_0^0(A,H)\right) + \xi g^2\left\{m_G^2 C^0 - D^0 - \frac{1}{2}\right\}$
$k)$	$2\cdot \xi g^2 \left\{D^0 + \frac{1}{2}\right\}$

Table 2:

$$B_0^1(1,2) \;=\; \frac{1}{2}\frac{m_1^2 + m_2^2}{(m_1^2 - m_2^2)^2} + \frac{m_1^2 m_2^2 \ln \dfrac{m_1^2}{m_2^2}}{(m_1^2 - m_2^2)^3}$$

where

$$\ln \mu^2 \equiv \frac{2}{\epsilon} - \gamma + \ln 4\pi\mu_0^2 \tag{90}$$

As a shorthand, we introduce

$$C^i = \frac{B_0^i(H,+) - B_0^i(H,-)}{m_+^2 - m_-^2}; \qquad D^i = \frac{m_+^2 B_0^i(H,+) - m_-^2 B_0^i(H,-)}{m_+^2 - m_-^2} \tag{91}$$

with m_+ and m_- defined by Eq.(85).

Tensor integrals were reduced to scalar integrals and expressed in terms of B_0^0 and B_0^1 with the help of covariant decomposition and contractions. An overall factor $\frac{i}{16\pi^2}$ is implicit throughout.From the Table 2, it is straightforward to check that $Z_1 = Z_3 = Z_4$.

$Z_\rho = Z_1 + Z_2$ is evaluated in a similar way, leading to the following cumbersome expression for the 1–loop correction (one can easily extract Z_2 from it since Z_1 is already known):

$$
\begin{aligned}
Z_{\rho,1}[\phi] \;=\; & -3g^2\left(\frac{m_+^2-\xi m_A^2}{m_+^2-m_-^2}B_0^0(A,+)\;+\;\frac{m_-^2-\xi m_A^2}{m_-^2-m_+^2}B_0^0(A,-)-1\right)+\\
+ \;\; & \xi g^2\left(B_0^0(+,-)+\frac{1}{3}\right)\;+\;\frac{\lambda^2\phi^2}{12\,m_H^2}\;+\;\frac{\lambda^2\phi^2}{108\,m_G^2}\;+\\
+ \;\; & \frac{\ln\dfrac{m_+^2}{m_-^2}}{(m_+^2-m_-^2)^3}\,\xi^2 m_A^2 m_G^2\left(4\xi g^2 m_A^2-2g^2 m_G^2+\frac{\lambda}{3}m_A^2\right)+\\
+ \;\; & 3\frac{\xi g^2 m_A^2}{m_+^2-m_-^2}\left(\frac{m_+^2\ln\dfrac{m_-^2}{m_A^2}}{m_A^2-m_-^2}\;-\;\frac{m_-^2\ln\dfrac{m_+^2}{m_A^2}}{m_A^2-m_+^2}\right)+\\
+ \;\; & 6\frac{\xi g^2 m_A^2}{m_+^2-m_-^2}\left(\frac{m_-^2\ln m_-^2-m_A^2\ln m_A^2}{m_A^2-m_-^2}\;-\;\frac{m_+^2\ln m_+^2-m_A^2\ln m_A^2}{m_A^2-m_+^2}+\right)\\
+ \;\; & \frac{\xi m_A^2}{(m_+^2-m_-^2)^2}\left(\frac{2}{3}\lambda\xi m_A^2\;+\;\frac{10}{3}\xi g^2 m_G^2\;+\;\frac{\lambda^2\phi^2}{108}\;-\;\frac{4}{9}\lambda m_G^2\right)+\\
+ \;\; & \frac{B_0^1(+,-)}{(m_+^2-m_-^2)^2}\xi^2 m_A^4\left(\frac{4}{3}\lambda m_G^2\;-\;\frac{\lambda^2\phi^2}{9}\;-\;16\xi g^2 m_G^2\right)
\end{aligned}
\tag{92}
$$

Once again, we want to emphasize that Z_2 does not vanish.

Appendix D

Shifting $\rho \to \rho + \upsilon$ in the Lagrangian (29) one gets the following Feynman rules in the polar gauge (neglecting trivial ghost fields):

Propagators:

ρ-field:	———	$\dfrac{i}{k^2 - m_H^2}$
Gauge:	$\mu \sim\sim\sim \nu$	$-i\left[\dfrac{1}{k^2 - m_A^2}(g^{\mu\nu} - \dfrac{k^\mu k^\nu}{k^2}) \quad + \quad \dfrac{\xi k^\mu k^\nu}{k^4}\right]$
χ-field:	- - - - - -	$i\,\dfrac{k^2 - \xi m_A^2}{k^4 \upsilon^2}$
mixed:	$\mu \sim\sim$ - - $\overset{k}{\to}$ -	$-\,\dfrac{\xi g k_\mu}{k^4}$

Vertices:

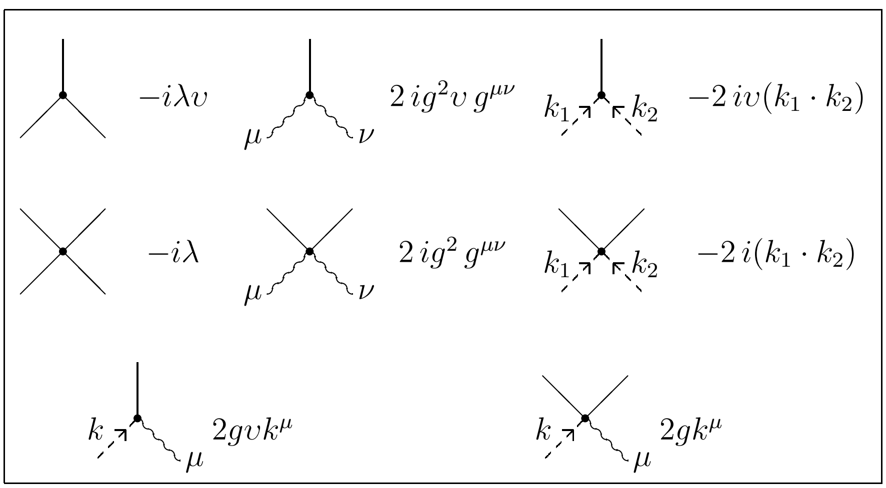


Table 3: Feynman rules in the polar gauge

In the unitary gauge the χ-field is absent. So the Feynman rules in the unitary gauge can be obtained from the above tables if we delete all propagators and vertices that involve χ and replace the gauge propagator with:

$$D^U_{\mu\nu}(k) = -\,\frac{i}{k^2 - m_A^2}\,(\,g^{\mu\nu} - \frac{k^\mu k^\nu}{m_A^2}) \tag{93}$$

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